Exam 1

1. Systems A and B consist of four linear equations in four unknowns modulo 2.

$$\begin{array}{|ccccc|}\hline x_1 & + & x_2 & = & 0 \\ + & & + & & \\ y_1 & + & y_2 & = & 0 \\ = & & = & & \\ 1 & & 1 & & \\ \hline\end{array}$$
system A

$$\begin{array}{|ccccc|}\hline x_1 & + & x_2 & = & 1 \\ + & & + & & \\ y_1 & + & y_2 & = & 0 \\ = & & = & & \\ 1 & & 1 & & \\ \hline\end{array}$$
system B

   (a) Solve system A using modular Gauss-Jordan elimination. If there are multiple solutions, specify the free variables and the assignment to the remaining variables in terms of the free variables.

   (b) In system B the value in the first equation was flipped. Produce a contradictory linear combination for system B. You may use any method you like.

2. You want to apply Gradient Descent to the system

$$\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}.$$

   The eigenvalues of the matrix are $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})$, $\mathbf{v}_2 = (1/\sqrt{2}, -1/\sqrt{2})$.

   (a) What is the maximum rate ρ^* below which convergence to the unique solution $(1/4, 1/4)$ is guaranteed (for any initialisation)?

   (b) You run Gradient Descent with rate $\rho = 0.01$ and initialisation $(x, y) = (1, 0)$. The distance between the state (x_t, y_t) at time t and the solution $(1/4, 1/4)$ is $\Theta(b^t)$ for some number b. Find b.

3. Calculate the Fourier transforms of the following functions. You may use any method you like. $+$ and $-$ are shorthand for the numbers $+1$ and -1.

   (a)

x_1x_2	$++$	$-+$	$+-$	$--$
$f(x_1x_2)$	1	1	-1	-1

   (b)

x_1x_2	$++$	$-+$	$+-$	$--$
$g(x_1x_2)$	0	1	1	1

   (**Hint:** What is $1 - g$?)

   (c) $h(x_1x_2x_3) = \begin{cases} f(x_1x_2), & \text{if } x_3 = -1 \\ g(x_1x_2), & \text{if } x_3 = +1. \end{cases}$

   (**Hint:** Apply one of the steps of the Fast Fourier-Walsh algorithm.)

Exam 2

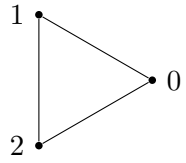
1. Let S be the system of equations

$$x + y + z = 0$$
$$x + 2y + 4z = 0$$

   (a) Apply Gauss-Jordan elmination to find a nonzero solution. Explain your steps.

   (b) Now add the equation $x + 3y + 7z = 1$ to the system. Find a contradictory linear combination.

2. Let A be the adjacency matrix of the 3-cycle (with zeros on the diagonal). The largest eigenvalue of A is $\lambda_1 = 2$. The corresponding eigenvector is $\mathbf{v}_1 = (1/\sqrt{3})(1, 1, 1)$.



   (a) Let $\mathbf{x}$ be any vector that is orthogonal to $\mathbf{v}_1$. Show that the state of Power Iteration on A initialized with $\mathbf{x}$ oscillates between $\mathbf{x}$ and $-\mathbf{x}$. Assume there are no precision errors.

   (b) Use part (a) to deduce the other two eigenvalues λ_2 and λ_3 of A. What are they?

3. Apply the modular Fast Fourier transform to calculate the polynomial representation of

x	1	i	-1	$-i$
$f(x)$	0	1	2	3

   (a) Calculate the list representations of f^+ and f^-, where $f(x) = f^+(x^2) + xf^-(x^2)$.

   (b) Calculate the polynomial representations of f^+ and f^- over domain $\{-1, +1\}$.

   (c) Calculate the polynomial representation of f.

Exam 3

1. You apply gradient descent to the following (inconsistent) univarate linear system:

$$x = 0$$
$$x = 1$$

   (a) Write the sum-of-squares loss f. For which value of $x^\star$ is $f(x^\star)$ minimized?

   (b) Let x_t be the state after t steps of gradient descent with rate ρ. What is $(x_t - x^\star)/(x_{t-1} - x^\star)$?

   (c) Assuming $x_0 = 0$ and $\rho = 1/8$, what is x_{20}?

2. You apply subspace iteration on the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

   Your initial state is the standard basis $\mathbf{x}_1 = (1, 0, 0)$, $\mathbf{x}_2 = (0, 1, 0)$, $\mathbf{x}_3 = (0, 0, 1)$.

   (a) Ignoring $\mathbf{x}_3$ for now, show how $\mathbf{x}_1$ and $\mathbf{x}_2$ evolve in the first three rounds of subspace iteration.

   (b) Use part (b) to find two of the three eigenvalues of A. Explain your reasoning.

   (c) Prove that if, at any point during power iteration, $A\mathbf{x}_i$ and $A\mathbf{x}_j$ become linearly dependent (for some $i \neq j$), zero must be an eigenvalue of A.

   (d) Use part (c) to find the remaining eigenvalue of A. (**Hint:** What happens to $\mathbf{x}_3$?)

3. The *2-bit addressing function* $\text{addr}\colon \{+1, -1\}^6 \to \{+1, -1\}$ is given by

$$\text{addr}(x, y, z_{++}, z_{+-}, z_{-+}, z_{--}) = z_{xy}.$$

   (a) Express addr in terms of the four point functions $\text{point}_{ab}(x, y)$ and the four variables z_{ab}.

   (b) Calculate $\widehat{\text{addr}}(\varnothing)$, $\widehat{\text{addr}}(\{1, 2, 3\})$ and $\widehat{\text{addr}}(\{1, 2, 3, 4\})$. (Use part (a).)